Problem 1. Potential games (6 points)

Consider a game between two players. Each player has two pure strategies A and B. Both players are **maximizers**. The payoff matrix is given as:

	A	B
A	$(3,3)$	$(0,5)$
B	$(5,0)$	$(2,2)$

In each entry, the first element denotes the utility of the row player and the second element denotes the utility of the column player. The above is a potential game.

a) For the function $\Phi : \{A, B\} \times \{A, B\} \to \mathbb{R}$ below, complete the two missing values $x_1, x_2 \in \mathbb{R}$ so that Φ will be a potential function for the game. (1 point)

$$\Phi = \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 0 & x_1 \\ 2 & x_2 \end{bmatrix} \end{array}$$

Solution: The solution is $x_1 = 2$ and $x_2 = 4$. This can be seen as follows: When the row player chooses action A and the column player changes from action A to B, her payoff increases by 2. Hence

$$x_1 = \Phi(A, B) = \Phi(A, A) + (J_2(A, B) - J_2(A, A)) = 0 + (5 - 3) = 2.$$

Similarly, when the row player chooses action B and the column player changes from action A to B, her payoff increases by 2. Therefore

$$x_2 = \Phi(B, B) = \Phi(B, A) + (J_2(B, B) - J_2(B, A)) = 2 + (2 - 0) = 4.$$

b) For a given game with N players, and player i's costs $J_i : \Gamma_1 \times \cdots \times \Gamma_N \to \mathbb{R}$ for $i = 1, \ldots, N$, show that if Φ and Ψ are both potential functions, then there exists a constant $c \in \mathbb{R}$ such that $\Phi(s) - \Psi(s) = c$, for every joint strategy $s \in \Gamma_1 \times \cdots \times \Gamma_N$. (2 points)

Solution: Let s be any joint strategy and consider some player i. By the potential property we have for any s_i',

$$J_i(s_i', s_{-i}) - J_i(s_i, s_{-i}) = \Phi(s_i', s_{-i}) - \Phi(s_i, s_{-i}),$$
$$J_i(s_i', s_{-i}) - J_i(s_i, s_{-i}) = \Psi(s_i', s_{-i}) - \Psi(s_i, s_{-i}).$$

Equating the right-hand sides and rearranging gives

$$\Phi(s_i, s_{-i}) - \Psi(s_i, s_{-i}) = \Phi(s_i', s_{-i}) - \Psi(s_i', s_{-i}).$$

Since we have chosen arbitrary s, i, and s_i', this shows that the difference $\Phi(s) - \Psi(s)$ is the same (i.e. equal to some constant $c \in \mathbb{R}$) for any joint strategy s.

c) Consider a student social network where each student communicates to their friends, and this connection is represented by a graph with N vertices and E edges. Namely, if student i and j are friends, then there is an edge between vertices i and j. On a given topic (such as if the course ME-429 is good or not), each student has an opinion, captured by $x_i \in \{0, 1\}$, $i = 1, \ldots, N$, where $0, 1$ represent disagree and agree, respectively. Each student feels some conformity to their friends and thus, tries to adjust her opinion by **minimizing** deviation of her opinion from those of her friends: $J_i(x) = \sum_{j \in \mathcal{N}_i} |x_i - x_j|$, where $\mathcal{N}_i$ denotes the set of friends of student i, and $x \in \{0, 1\}^N$.

i. Show that the game characterized by $\{J_i\}_{i=1}^N$ is a potential game. Hint: verify $\frac{1}{2} \sum_{i=1}^N J_i(x)$ is an exact potential function for the game. (1 point)

Solution: We can verify that $\phi(x) = \frac{1}{2} \sum_{i=1}^N J_i(x) = \frac{1}{2} \sum_{i=1}^N \sum_{j \in N_i} |x_i - x_j|$, a function that counts the number of edges on which players disagree, is a potential function. Indeed, note that

$$\begin{aligned} \phi(x_i', x_{-i}) - \phi(x_i'', x_{-i}) &= \frac{1}{2} \sum_{k=1}^N J_k(x_i', x_{-i}) - \frac{1}{2} \sum_{k=1}^N J_k(x_i'', x_{-i}) \\ &= \sum_{j \in \mathcal{N}_i} |x_i' - x_j| - \sum_{j \in \mathcal{N}_i} |x_i'' - x_j| \\ &= J_i(x_i', x_{-i}) - J_i(x_i'', x_{-i}). \end{aligned}$$

ii. Determine two Nash equilibria of the game above. (1 point)
Solution: Observe that that all students agreeing is a minimizer of the potential function, as for such x, $\phi(x) = 0$ *(and clearly* $\phi(x) \geq 0$ *for all x). We know that in potential games, when players are minimizers, the minimizer of the potential function corresponds to a Nash equilibrium.*

iii. Show that the best response of a student to the neighbors' strategies $\{x_j\}_{j \in \mathcal{N}_i}$ is the majority opinion of her neighbors. (1 point)
Solution: For fixed $\{x_j\}_{j \in \mathcal{N}_i}$, $\min_{x_i} J_i(x_i, x_{-i}) = \min_{x_i} \sum_{j \in \mathcal{N}_i} |x_i - x_j|$ *by definition corresponds to choosing* x_i *as the majority of its neighbors' strategies (where ties can be broken arbitrarily).*

Problem 2. Zero-Sum game (7 points)

Consider a zero-sum game between two players captured by matrix A below. The minimizer chooses a row and the maximizer chooses a column.

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 0 & -2 \\ -3 & 2 & 0 \end{bmatrix}.$$

a) Identify any rows or columns that are strictly dominated. Iteratively eliminate strictly dominated rows and columns. (2 points)

Solution:

Step 1 (Strictly dominated Row): the minimizer chooses among the rows, thus a row is strictly dominated if there is another row always giving a smaller payoff. Row 1 is strictly dominated by Row 2 (since each entry in Row 2 is strictly smaller than the respecting entry of Row 1). Removing Row 1, we have:

$$\begin{bmatrix} 2 & 0 & -2 \\ -3 & 2 & 0 \end{bmatrix}$$

Step 2 (Strictly dominated Column): for the maximizer, a column is strictly dominated if there is another column giving a larger payoff. Column 3 is strictly dominated by Column 2. Removing Column 3, we have:

$$\begin{bmatrix} 2 & 0 \\ -3 & 2 \end{bmatrix}$$

b) Consider now a zero-sum game with utilities captured by the matrix $B \in \mathbb{R}^{2\times 2}$ below. Consider the strategies $y^* = \left[\frac{1}{3}, \frac{2}{3}\right]$ for the minimizer (row player) and $z^* = \left[\frac{1}{6}, \frac{5}{6}\right]$ for the maximizer (column player). Verify that (y^*, z^*) is a saddle point equilibrium. (2 points)

$$B = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$$

Solution:

We need to verify that $(y^*)^\top B z^* \leq y^\top B z^*$ for any y and $(y^*)^\top B z^* \geq (y^*)^\top B z$ for any z . It is sufficient to see if the payoff is smaller than or equal to any pure strategy for the minimizer, and bigger than or equal to any pure strategy for the maximizer, as we proved in Slide 21 of Lecture 2. Let's consider first the minimizer:

$$Bz^* = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{6} \\ \frac{5}{6} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}.$$

Observe that for any pure strategy ($y = \begin{bmatrix} 1, & 0 \end{bmatrix}$ or $y = \begin{bmatrix} 0, & 1 \end{bmatrix}$) she would still get $\frac{1}{3}$. Similarly, for the maximizer

$$(y^*)^\top B = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

Again, no matters which pure strategy ($z = \begin{bmatrix}1, & 0\end{bmatrix}$ or $z = \begin{bmatrix}0, & 1\end{bmatrix}$) she chooses, the payoff is always $\frac{1}{3}$.

c) Consider a 2-player zero-sum game, played sequentially and under the full information feedback. Let Alice be the minimizer and Bob be the maximizer. The cost for Alice is captured by a matrix $C \in \mathbb{R}^{m \times n}$.

i. Consider Alice playing first. Show that Alice's strategy arising from backward induction corresponds to her security strategy. (1 point)

Solution: With backward induction, Bob will choose to maximize between her n actions, for every m actions of Alice. Hence, Bob will choose strategy $\arg\max_{j \in \{1,\dots,n\}} c_{ij}$ for every $i = 1, \dots, m$ information sets. The entry corresponding to information set i for $i = 1, 2, \dots, n$, would be $\max_{j \in \{1,\dots,n\}} c_{ij}$. Thus, Alice will then do $\min_{i \in \{1,\dots,m\}} \max_{j \in \{1,\dots,n\}} c_{ij}$. Note that this corresponds to Alice's security value, and the resulting i^* would be Alice's security strategy.

ii. Based on the above and your knowledge of the min-max inequality, argue why Alice would prefer to go second. (1 point)

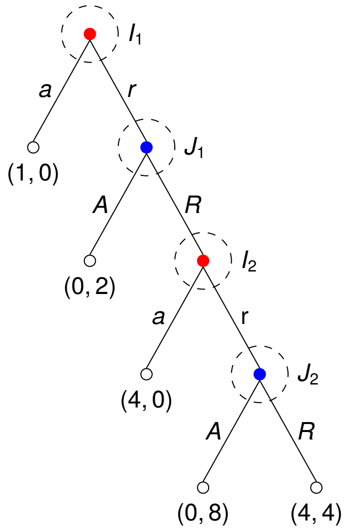
Solution: Note that if Alice goes first, she would get $\min_i \max_j c_{ij}$ and if she goes second, she would get $\max_j \min_i c_{ij}$. From min-max inequality we know $\min_i \max_j c_{ij} \geq \max_j \min_i c_{ij}$.

iii. Under which condition Alice would be indifferent to going first or second? (1 point)

Solution: In case there exists a pure Nash equilibrium for the game then, $\min_i \max_j c_{ij} = \max_j \min_i c_{ij}$ and, thus, going first or second will give the same outcome.

Problem 3. Backward induction in extensive form games (7 points)

Consider the multi-stage game shown in the figure on the right. There is a pot of money, starting with 1 CHF. At each iteration, the decision-maker (player 1 or player 2) can decide to accept the pot ending up with the outcome shown in the left leaf of the tree, or to refuse the pot, at which point the pot size doubles and the next player makes the same decision.



Game tree
(red: player 1, blue: player 2)

a) Is this a feedback game? Is this a full information game? (1 point)

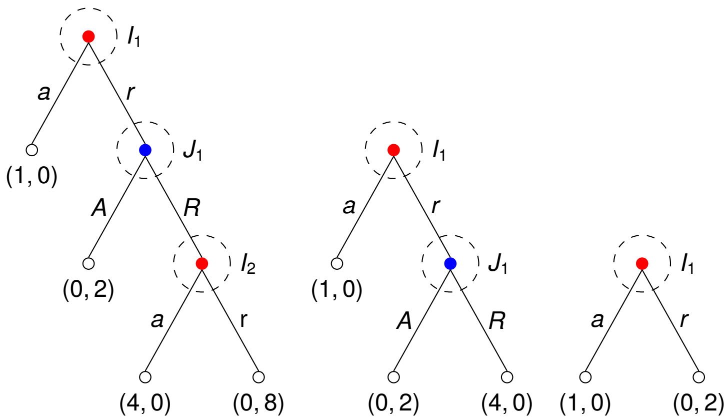
   Solution: Yes, yes.

b) Given the information sets shown, how many strategies does each player have? (1 point)

   Solution: Each player has two information sets and each information set has two possible actions. Hence, there are 4 strategies (2 actions per 2 strategies)

c) Use backward induction to compute the subgame perfect Nash equilibrium strategy and the outcome of the game. Show your work by drawing the game tree on your solution sheet and showing the backward induction. (2.5 points)

   Solution: At each info set, the players should accept, hence the outcome is $(1, 0)$. On the game tree, this looks as follows:



d) Consider the social welfare function of maximizing the sum of both players' payoffs $C : \Gamma_1 \times \Gamma_2 \to \mathbb{R}$. What would be the players' strategies maximizing the social welfare and the corresponding social welfare value? (2 points)

   Solution: Reject at each iteration resulting in $(4, 4)$, or reject at each iteration except the last one resulting in $(0, 8)$. Hence the corresponding social welfare value is 8.

e) The price of anarchy is defined as $\frac{C(s^{\text{opt}})}{\min_{s \in S^{\text{NE}}} C(s)}$, where $s^{opt} \in \Gamma_1 \times \Gamma_2$ is a socially optimal strategy and the minimum in the denominator is taken over Nash equilibrium strategies, S^{NE}. What is the price of anarchy in this game? (0.5 point)

Solution: Observe that backward induction gives the outcome of $(1, 0)$ with a total payoff of both players as $1+0 = 1$. Furthermore, since it is a game with perfect information, there will always be a pure strategy Nash equilibrium, implying that the above equilibrium has the worst social welfare value. In the previous subtask, we have seen that the optimal social welfare value is 8. Therefore, the price of anarchy is $8/1 = 8$.